ON THE STRUCTURE OF THE ELASTIC TENSOR AND THE CLASSIFICATION OF ANISOTROPIC MATERIALS

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In the linear theory of elasticity the deformation energy density for anisotropic materials has the form [1-3]

$$2\Phi = A_{ijkl}\varepsilon_{ij}\varepsilon_{kl}, \tag{1}$$

where $\varepsilon_{ij} = \varepsilon_{ji}$ is the deformation tensor in the orthogonal coordinate system x_1, x_2, x_3 and $A_{ijk\ell}$ are the components of the modulus of elasticity tensor. In (1) and below, repeating indices denote a summation from one to three. The constants $A_{ijk\ell}$ have the symmetry properties:

$$A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij}, \tag{2}$$

which follows from the symmetry of the tensor ε_{ij} and the possibility of redefining the summation indices in (1). We see from (2) that there are only 21 independent components of $A_{ijk\ell}$. The deformation energy (1) must be a positive definite quadratic form [1-3].

The stress tensor is determined from (1) according to the equation

$$\sigma_{ij} = \partial\Phi/\partial\varepsilon_{ij} = A_{ijkl}\varepsilon_{kl}. \tag{3}$$

Equation (3), the so-called generalized Hooke's law, can be inverted:

$$\varepsilon_{ij} = a_{ijkl}\sigma_{kl}. \tag{4}$$

Here $a_{ijk\ell}$ is the compliance tensor. The constants $a_{ijk\ell}$ satisfy the symmetry conditions (2) and are related to the $A_{ijk\ell}$ by

$$A_{ijkl}a_{klrs} = \delta_{ijrs} = \frac{1}{2}(\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}),$$
$$a_{ijkl}A_{klrs} = \delta_{ijrs},$$

where $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$. The tensor δ_{ijrs} plays the role of a unit tensor in the space of symmetric tensors of the form (2).

For an orthogonal coordinate transformation

$$x_i = c_{ij}x'_j,\ x'_j = c_{ij}x_i,\ c_{ij}c_{kj} = \delta_{ik} \tag{5}$$

the tensors ε_{ij}, $A_{ijk\ell}$ transform as

$$\varepsilon_{ij} = c_{ik}c_{jl}\varepsilon'_{kl},\ \varepsilon'_{kl} = c_{ik}c_{jl}\varepsilon_{ij}, \tag{6}$$
$$A_{ijkl} = c_{ip}c_{jq}c_{kr}c_{ls}A'_{pqrs},$$
$$A'_{pqrs} = c_{ip}c_{jq}c_{kr}c_{ls}A_{ijkl}.$$

Because of the choice of three free parameters of c_{ij} as determining the position of the coordinate system (5), the number of independent components of $A_{ijk\ell}$, characterizing the elastic properties of the material, decreases from 21 to 18 [4]. When there are various symmetries in the structure of anisotropic materials, the number of independent components of $A_{ijk\ell}$ is still smaller [1-4].

The representation of Hooke's law (3) in special bases and the range of variation of the constants $A_{ijk\ell}$ consistent with the positive definiteness of the quadratic form (1)

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have been considered in [3-8].

In the present paper we reduce the quadratic form (1) to canonical form, which reveals the structure of the constants $A_{ijk\ell}$. We also give a new classification of anisotropic materials.

We consider six deformation tensors t_{ijpq}. Here the first two indices denote the components of the tensor, and the last two give the number of the tensor, where tensors with numbers (pq) and (gp) are identical. Therefore the tensor t_{ijpq} is symmetric with respect to each pair of indices:

$$t_{ijpq} = t_{jipq},\ t_{ijpq} = t_{ijqp},$$

and hence there are only 36 independent quantities t_{ijpq}.

Consider now the expression

$$\tilde{A}_{pqrs} = A_{ijkl} t_{ijpq} t_{klrs}. \tag{7}$$

By the definition of the tensor $A_{ijk\ell}$ in (3), the expression $A_{ijk\ell} t_{k\ell rs}$ represents the components of the stress tensor σ_{ijrs}. Further, this tensor contracts with the tensor t_{ijpq}:

$$A_{ijkl} t_{klrs} t_{ijpq} = \sigma_{ijrs} t_{ijpq}.$$

Similarly (in view of the symmetry of the tensor $A_{ijk\ell}$) we have

$$A_{ijkl} t_{ijpq} t_{klrs} = \sigma_{klpq} t_{klrs}.$$

Hence the quantities (7) represent the contraction of the corresponding stress and deformation tensors. They are scalar and therefore are invariant to orthogonal coordinate transformations of the type (5).

We choose the tensors t_{ijpq} such that

$$t_{ijpq} t_{ijrs} = \delta_{pqrs}; \tag{8}$$

$$A_{ijkl} t_{ijpq} t_{klrs} = 0,\ (pq) \neq (rs). \tag{9}$$

The condition (8) implies the orthonormality (in the sense of contraction) of the tensors t_{ijpq}, and also implies the orthogonality of the matrices t_{ijrs}. Condition (9) means that the stress tensors σ_{ijrs}, $\sigma_{k\ell pq}$ correspond to the deformation tensors $t_{k\ell rs}$, t_{ijpq} and are proportional to them. Equation (8) contains 21 equations, and (9) contains 15, and therefore we have 36 equations (8) and (9) for the 36 independent quantities t_{ijpq}. It follows from (9) that

$$\tilde{A}_{pqrs} = 0,\ (pq) \neq (rs). \tag{10}$$

We multiply both sides of (7) by t_{mnpq}, t_{fgrs} and sum over p, q, r, s:

$$A_{ijkl} t_{ijpq} t_{klrs} t_{mnpq} t_{fgrs} = \tilde{A}_{pqrs} t_{mnpq} t_{fgrs}. \tag{11}$$

It follows from (8) that

$$t_{ijpq} t_{mnpq} = \delta_{ijmn},\ t_{klrs} t_{fgrs} = \delta_{klfg}.$$

Now (11) takes the form

$$A_{ijkl} \delta_{ijmn} \delta_{klfg} = A_{ijfg} \delta_{ijmn} = A_{mnfg} = \tilde{A}_{pqrs} t_{mnpq} t_{fgrs},$$

or, replacing the indices mnfg by ijkℓ, we obtain

$$A_{ijkl} = \tilde{A}_{pqrs} t_{ijpq} t_{klrs},\ (pq) = (rs). \tag{12}$$

Here we take into account (10) in the summation. Therefore if the tensor $A_{ijk\ell}$ is given, one can determine t_{ijpq} from (8) and (9), and then find $\tilde{A}_{pqrs}$ from (7) ((pq) = (rs)). If we are given the six numbers $\tilde{A}_{pqrs}$, (pq) = (rs), and the 36 quantities t_{ijpq}, connected by the 21 relations (8), then from (12) we can construct the tensor $A_{ijk\ell}$, which depends on the six quantities $\tilde{A}_{pqrs}$ and the 15 parameters of t_{ijpq}, which remain free parameters after imposing the conditions (8).

Equations (8) and (9) are invariant with respect to orthogonal coordinate transformations (5). Equation (12) also does not change in form:

$$A'_{ijkl} = \tilde{A}_{pqrs} t'_{ijpq} t'_{klrs}, \quad (pq) = (rs)$$

(quantities with primes are defined by (6)).

Using (12), we find the stress tensor corresponding to the deformation tensor $t_{k\ell mn}$:

$$A_{ijkl} t_{klmn} = \tilde{A}_{pqrs} t_{ijpq} t_{klrs} t_{klmn} = \tilde{A}_{pqrs} t_{ijpq} \delta_{rsmn} = \tilde{A}_{pqmn} t_{ijpq} = \begin{cases} \tilde{A}_{mnmn} t_{ijmn} & \text{for} \quad m = n, \\ 2\tilde{A}_{mnmn} t_{ijmn} & \text{for} \quad m \neq n \end{cases} \tag{13}$$

(summation is not carried out over m and n). We see from (13) that it is proportional to the deformation tensor:

$$A_{ijkl} t_{klmn} = \lambda t_{ijmn}, \tag{14}$$

and the proportionality coefficients are $\lambda = \tilde{A}_{mnmn}$ $(m = n)$ and $\lambda = 2\tilde{A}_{mnmn}$ $(m \neq n)$. We rewrite (14) in the form

$$(A_{ijkl} - \lambda \delta_{ijkl}) t_{klmn} = 0. \tag{15}$$

If (15) is regarded as a system of homogeneous linear equations for the $t_{k\ell mn}$, then this system will have a nonzero solution when its determinant vanishes [9]:

$$|A_{ijkl} - \lambda \delta_{ijkl}| = 0. \tag{16}$$

Because the number of independent equations in (15) is only six, and the matrix of the coefficients of (15) is symmetric, then the 9th-order determinant (16) will have identical rows and columns with indices (ij), (ji), i ≠ j, (kℓ), (ℓk), k ≠ ℓ, and therefore the 9th-order determinant (16) is identically zero. The determinant (16) can be considered as a 6th-order determinant, where the rows and columns with indices (ji) = (ℓk), i ≠ j are eliminated. In result we obtain a 6th-order determinant whose corresponding matrix is also symmetric, and a 6th-order equation for λ, which has six real roots [9].

And so (16) is written in the form

$$\begin{vmatrix} A_{11}^{11} - \lambda & A_{22}^{11} & A_{33}^{11} & \sqrt{2} A_{23}^{11} & \sqrt{2} A_{13}^{11} & \sqrt{2} A_{12}^{11} \\ A_{11}^{22} & A_{22}^{22} - \lambda & A_{33}^{22} & \sqrt{2} A_{23}^{22} & \sqrt{2} A_{13}^{22} & \sqrt{2} A_{12}^{22} \\ A_{11}^{33} & A_{22}^{33} & A_{33}^{33} - \lambda & \sqrt{2} A_{23}^{33} & \sqrt{2} A_{13}^{33} & \sqrt{2} A_{12}^{33} \\ \sqrt{2} A_{11}^{23} & \sqrt{2} A_{22}^{23} & \sqrt{2} A_{33}^{23} & 2A_{23}^{23} - \lambda & 2A_{13}^{23} & 2A_{12}^{23} \\ \sqrt{2} A_{11}^{13} & \sqrt{2} A_{22}^{13} & \sqrt{2} A_{33}^{13} & 2A_{23}^{13} & 2A_{13}^{13} - \lambda & 2A_{12}^{13} \\ \sqrt{2} A_{11}^{12} & \sqrt{2} A_{22}^{12} & \sqrt{2} A_{33}^{12} & 2A_{23}^{12} & 2A_{13}^{12} & 2A_{12}^{12} - \lambda \end{vmatrix} = 0, \tag{17}$$

where $A_{k\ell}^{ij} = A_{ijk\ell}$. The elements of the matrix corresponding to (17) can be denoted as A_{ij}, where i, j now go from 1 to 6. The correspondence of the indices is obvious from (17). Expanding the determinant (17), we obtain a 6th-order equation for λ:

$$\lambda^6 - I_1 \lambda^5 + I_2 \lambda^4 - I_3 \lambda^3 + I_4 \lambda^2 - I_5 \lambda + I_6 = 0, \tag{18}$$

where the coefficients I_k $(k = \overline{1, 6})$ are invariants of the elastic tensor $A_{ijk\ell}$ and are given by the formulas [10, 11]

$$I_k = \frac{1}{k!} \begin{vmatrix} s_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ s_2 & s_1 & 2 & 0 & \cdot & \cdot & 0 \\ s_3 & s_2 & s_1 & 3 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_k & s_{k-1} & \cdot & \cdot & \cdot & \cdot & s_1 \end{vmatrix}, \quad k = \overline{1, 6},$$

$$s_1 = A_{ii}, \quad s_2 = A_{ij} A_{ji}, \quad s_3 = A_{ij} A_{jk} A_{ki},$$

$$s_4 = A_{ij} A_{jk} A_{kl} A_{li}, \quad s_5 = A_{ij} A_{jk} A_{kl} A_{lm} A_{mi},$$

$$s_6 = A_{ij} A_{jk} A_{kl} A_{lm} A_{mn} A_{ni}.$$

Here summation over repeating indices goes from 1 to 6. The nonzero solutions of (15) corresponding to the roots of (18) are orthonormal [9], i.e., they satisfy the conditions (8).

Hence it has been shown that there are two possibilities for determining the eigenvalues and eigentensors of the linear transformation (3): 1) the eigentensors t_{ijpq} are determined from (8) and (9), and then the eigenvalues $\tilde{A}_{pqrs}$, (pg) = (rs), are determined from (7); 2) from the characteristic equation (18) we find the six eigenvalues $\lambda = \tilde{A}_{mnmn}$ for $m = n$, $\lambda = 2\tilde{A}_{mnmn}$ for $m \neq n$, then for each root λ the nonzero solution of the system (15) is determined such that the orthonormality condition (8) is satisfied.

We see from (7) that the eigenvalues $\tilde{A}_{pqrs}$ represent deformation energies corresponding to the deformation tensor t_{ijpq}, (pq) = (rs). But the deformation energy is a positive definite quadratic form, and therefore is positive for any nonzero deformation tensor. Because the tensors t_{ijpq} are nonzero, the eigenvalues must be positive:

$$\tilde{A}_{pqrs} > 0,\ (pq) = (rs). \tag{19}$$

In the deformation energy (1) we substitute (12) for the coefficients $A_{ijk\ell}$:

$$2\Phi = A_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \tilde{A}_{pqrs}t_{ijpq}t_{klrs}\varepsilon_{ij}\varepsilon_{kl}. \tag{20}$$

In (20) we introduce the notation

$$\tilde{\varepsilon}_{pq} = t_{ijpq}\varepsilon_{ij},\ \tilde{\varepsilon}_{rs} = t_{klrs}\varepsilon_{kl}. \tag{21}$$

With (21) the deformation energy (20) can be written as

$$2\Phi = A_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \tilde{A}_{pqrs}\tilde{\varepsilon}_{pq}\tilde{\varepsilon}_{rs},\ (pq) = (rs). \tag{22}$$

From (22) we see that the deformation energy is a sum of squares of the variables $\tilde{\varepsilon}_{pq}$, (pq) = (rs), with positive coefficients. Therefore (21) is an orthogonal transformation of the variables ε_{ij} to the variables $\tilde{\varepsilon}_{pq}$ (the transformation matrix t_{ijpq} is orthogonal in view of (8)). This transformation brings the deformation energy (1) to canonical form (22). In order for a quadratic form to be positive definite it is necessary and sufficient that all of the eigenvalues of its matrix be positive [10]. Therefore the deformation energy (22) is a positive definite quadratic form because in the case considered here we have the conditions (19).

In view of the orthogonality of the matrix t_{ijpq}, the inverse of the transformation (21) is

$$\varepsilon_{ij} = t_{ijpq}\tilde{\varepsilon}_{pq}. \tag{23}$$

Equation (21) shows that the variable $\tilde{\varepsilon}_{pq}$ is the contraction of the two tensors t_{ijpq} and ε_{ij}, i.e., it is an invariant (scalar) with respect to orthogonal coordinate transformations (5). From (23) it is evident that the $\tilde{\varepsilon}_{pq}$ are the expansion coefficients of the tensor ε_{ij} with respect to the eigentensors t_{ijpq}.

And so it has been shown that independently of the choice of the orthogonal coordinate system, the deformation energy has the form

$$2\Phi = \tilde{A}_{pqrs}\tilde{\varepsilon}_{pq}\tilde{\varepsilon}_{rs},\ (pq) = (rs) \tag{24}$$

and is determined by 12 invariant quantities: the six eigenvalues $\tilde{A}_{pqrs}$, (pq) = (rs), and the six variables $\tilde{\varepsilon}_{pq}$, (pq) = (qp). The quantities $\tilde{\varepsilon}_{pq}$ depend on the eigentensor t_{ijpq} and the deformation tensor ε_{ij} (see (21)) and are arbitrary, in view of the arbitrariness of the deformation tensor ε_{ij}. For fixed $\tilde{\varepsilon}_{pq}$ (for example, two materials with identical values) anisotropic materials will be distinguished only by the values $\tilde{A}_{pqrs}$, (pq) = (rs). Hence the deformation energy (24) of the anisotropic material is completely characterized by the six quantities $\tilde{A}_{pqrs}$, (pq) = (rs); they do not depend on the choice of orthogonal coordinate system and can be called the intrinsic elastic moduli.

We write Hooke's law (3) in invariant form. Substitute (12) into (3):

$$\sigma_{mn} = \tilde{A}_{pqrs} t_{mnpq} t_{klrs} \varepsilon_{kl} = \tilde{A}_{pqrs} t_{mnpq} \tilde{\varepsilon}_{rs}. \tag{25}$$

We multiply (25) by t_{mnij} and sum with respect to m and n:

$$t_{mnij}\sigma_{mn} = \tilde{A}_{pqrs} t_{mnij} t_{mnpq} \tilde{\varepsilon}_{rs} = \tilde{A}_{pqrs}\delta_{ijpq}\tilde{\varepsilon}_{rs} = \tilde{A}_{ijrs}\tilde{\varepsilon}_{rs}. \tag{26}$$

On the left-hand side of (26) let

$$\tilde{\sigma}_{ij} = t_{mnij}\sigma_{mn}. \tag{27}$$

The transformation (27) corresponds completely to the relations (21). The inverse transformation to (27) is analogous to (23):

$$\sigma_{mn} = t_{mnij}\tilde{\sigma}_{ij}.$$

Using (27) we obtain from (26) Hooke's law in invariant form, independent of the choice of orthogonal coordinate system:

$$\tilde{\sigma}_{ij} = \tilde{A}_{ijrs}\tilde{\varepsilon}_{rs},\ (rs) = (ij). \tag{28}$$

We write out (28) as:

$$\begin{aligned}
&\tilde{\sigma}_{11} = \tilde{A}_{1111}\tilde{\varepsilon}_{11},\ \tilde{\sigma}_{12} = \tilde{A}_{1212}\tilde{\varepsilon}_{12} + \tilde{A}_{1221}\tilde{\varepsilon}_{21},\\
&\tilde{\sigma}_{13} = \tilde{A}_{1313}\tilde{\varepsilon}_{13} + \tilde{A}_{1331}\tilde{\varepsilon}_{31},\\
&\tilde{\sigma}_{21} = \tilde{A}_{2112}\tilde{\varepsilon}_{12} + \tilde{A}_{2121}\tilde{\varepsilon}_{21},\ \tilde{\sigma}_{22} = \tilde{A}_{2222}\tilde{\varepsilon}_{22},\\
&\tilde{\sigma}_{23} = \tilde{A}_{2323}\tilde{\varepsilon}_{23} + \tilde{A}_{2332}\tilde{\varepsilon}_{32},\ \tilde{\sigma}_{31} = \tilde{A}_{3113}\tilde{\varepsilon}_{13} + \tilde{A}_{3131}\tilde{\varepsilon}_{31},\\
&\tilde{\sigma}_{32} = \tilde{A}_{3223}\tilde{\varepsilon}_{23} + \tilde{A}_{3232}\tilde{\varepsilon}_{32},\ \tilde{\sigma}_{33} = \tilde{A}_{3333}\tilde{\varepsilon}_{33}.
\end{aligned}$$

Hooke's law (28) can be inverted to give:

$$\tilde{\varepsilon}_{rs} = \tilde{a}_{rskl}\tilde{\sigma}_{kl},\ (kl) = (rs); \tag{29}$$

$$\tilde{A}_{ijrs}\tilde{a}_{rskl} = \delta_{ijkl},\ (rs) = (ij),\ (kl) = (rs); \tag{30}$$

$$\begin{aligned}
&\tilde{a}_{1111} = \frac{1}{\tilde{A}_{1111}},\ \tilde{a}_{2222} = \frac{1}{\tilde{A}_{2222}},\ \tilde{a}_{3333} = \frac{1}{\tilde{A}_{3333}},\\
&2\tilde{a}_{2323} = \frac{1}{2\tilde{A}_{2323}},\ 2\tilde{a}_{1313} = \frac{1}{2\tilde{A}_{1313}},\ 2\tilde{a}_{1212} = \frac{1}{2\tilde{A}_{1212}}.
\end{aligned} \tag{31}$$

We multiply (29) by t_{ijrs} and sum with respect to r,s:

$$t_{ijrs}\tilde{\varepsilon}_{rs} = \tilde{a}_{rskl} t_{ijrs}\tilde{\sigma}_{kl}. \tag{32}$$

With (23) and (27) we can write (32) in the form

$$\varepsilon_{ij} = \tilde{a}_{rskl} t_{ijrs} t_{mnkl}\sigma_{mn}. \tag{33}$$

Comparing (33) with (4) we obtain

$$a_{ijmn} = \tilde{a}_{rskl} t_{ijrs} t_{mnkl},\ (rs) = (kl). \tag{34}$$

Hence the matrix (34) is the inverse of the matrix $A_{ijk\ell}$ of (12) and $\tilde{a}_{rsk\ell}$ is related to $\tilde{A}_{pqrs}$ by (30) or (31).

Because the intrinsic elastic moduli $\tilde{A}_{pqrs}$, (pq) = (rs) (the roots of equation (18)), can be enumerated arbitrarily, we adopt the following convention

$$\begin{aligned}
&\tilde{A}_{1111} \geqslant \tilde{A}_{2222} \geqslant \tilde{A}_{3333} \geqslant 2\tilde{A}_{2323} \geqslant 2\tilde{A}_{1313} \geqslant 2\tilde{A}_{1212} > 0,\\
&\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5 \geqslant \lambda_6 > 0.
\end{aligned} \tag{35}$$

The correspondence of the notations in (35) is obvious.

Using the second notation of (35), we write out the components of the modulus of elasticity tensor (12) and the coefficients of compliance tensor (34):

$$A_{ijkl} = \lambda_1 t_{ij11} t_{kl11} + \lambda_2 t_{ij22} t_{kl22} + \lambda_3 t_{ij33} t_{kl33} + \\ + \lambda_4 (t_{ij23} t_{kl23} + t_{ij32} t_{kl32}) + \lambda_5 (t_{ij13} t_{kl13} + t_{ij31} t_{kl31}) + \\ + \lambda_6 (t_{ij12} t_{kl12} + t_{ij21} t_{kl21}), \tag{36}$$

$$a_{ijkl} = \frac{1}{\lambda_1} t_{ij11} t_{kl11} + \frac{1}{\lambda_2} t_{ij22} t_{kl22} + \frac{1}{\lambda_3} t_{ij33} t_{kl33} + \frac{1}{\lambda_4} (t_{ij23} t_{kl23} + t_{ij32} t_{kl32}) + \\ + \frac{1}{\lambda_5} (t_{ij13} t_{kl13} + t_{ij31} t_{kl31}) + \frac{1}{\lambda_6} (t_{ij12} t_{kl12} + t_{ij21} t_{kl21}).$$

And so for any material the modulus of elasticity tensor and the compliance tensor are given by (36), (8), and (35), and based on this, it is possible to classify anisotropic materials according to the number of different moduli λ_k and their multiplicity.

For each material we set up the symbol $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, where $k \leq 6$, $\alpha_k \geq 1$, $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 6$. Here k is the number of different eigenvalues λ_i, and α_i is their multiplicity. The materials are classified into groups (classes) according to the number of different eigenvalues λ_i. The total number of groups is six, and they are subdivided into subclasses depending on the multiplicity of the eigenvalues. We write out for these groups and subclasses their symbols and the relations between the eigenvalues (35):

I. $\{6\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$;

II. 1. $\{1, 5\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$,

2. $\{2, 4\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$,

3. $\{3, 3\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 = \lambda_5 = \lambda_6$,

4. $\{4, 2\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 > \lambda_5 = \lambda_6$,

5. $\{5, 1\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 > \lambda_6$;

III. 1. $\{1, 1, 4\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6$,

2. $\{1, 2, 3\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 > \lambda_4 = \lambda_5 = \lambda_6$,

3. $\{1, 3, 2\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 > \lambda_5 = \lambda_6$,

4. $\{1, 4, 1\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 > \lambda_6$,

5. $\{2, 1, 3\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 = \lambda_5 = \lambda_6$,

6. $\{2, 2, 2\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > \lambda_5 = \lambda_6$,

7. $\{2, 3, 1\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 = \lambda_5 > \lambda_6$,

8. $\{3, 1, 2\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 > \lambda_5 = \lambda_6$,

9. $\{3, 2, 1\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 = \lambda_5 > \lambda_6$,

10. $\{4, 1, 1\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 > \lambda_5 > \lambda_6$;

IV. 1. $\{1, 1, 1, 3\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 = \lambda_5 = \lambda_6$,

2. $\{1, 1, 2, 2\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 > \lambda_5 = \lambda_6$,

3. $\{1, 1, 3, 1\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 = \lambda_5 > \lambda_6$,

4. $\{1, 2, 1, 2\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 > \lambda_4 > \lambda_5 = \lambda_6$,

5. $\{1, 2, 2, 1\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 > \lambda_4 = \lambda_5 > \lambda_6$,

6. $\{1, 3, 1, 1\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 = \lambda_4 > \lambda_5 > \lambda_6$,

7. $\{2, 1, 1, 2\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 = \lambda_6$,

8. $\{2, 1, 2, 1\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 = \lambda_5 > \lambda_6$,

9. $\{2, 2, 1, 1\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 = \lambda_4 > \lambda_5 > \lambda_6$,

10. $\{3, 1, 1, 1\} \leftrightarrow \lambda_1 = \lambda_2 = \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6$;

V. 1. $\{1, 1, 1, 1, 2\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 = \lambda_6$,

2. $\{1, 1, 1, 2, 1\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 = \lambda_5 > \lambda_6$,

3. $\{1, 1, 2, 1, 1\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 = \lambda_4 > \lambda_5 > \lambda_6$,

4. $\{1, 2, 1, 1, 1\} \leftrightarrow \lambda_1 > \lambda_2 = \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6$,

5. $\{2, 1, 1, 1, 1\} \leftrightarrow \lambda_1 = \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6$;

VI. $\{1, 1, 1, 1, 1, 1\} \leftrightarrow \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > \lambda_5 > \lambda_6$.

It is evident from these relations that all materials can be classified into 32 classes (1 + 5 + 10 + 10 + 5 + 1 = 32) and each to class there uniquely corresponds the symbol $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ which characterizes the structure of the material. The order of the numbers in the symbols is significant; upon permuting the number in the symbol we obtain a material of a different class, with a different internal structure.

A more detailed classification of anisotropic materials can be carried out according to the form of the eigentensors t_{ijpq}.

For materials with the symbols {6}, {1, 5}, {5, 1} the elastic moduli (36) take the form

$$A_{ijkl} = \lambda_1 \delta_{ijkl}; \tag{37}$$

$$A_{ijkl} = (\lambda_1 - \lambda_2) t_{ij11} t_{kl11} + \lambda_2 \delta_{ijkl}; \tag{38}$$

$$A_{ijkl} = \lambda_1 \delta_{ijkl} - (\lambda_1 - \lambda_6) 2 t_{ij12} t_{kl12}; \tag{39}$$

and in view of (8)

$$t_{ij11} t_{ij11} = 1, \; t_{ij12} t_{ij12} = 1/2. \tag{40}$$

Materials with elastic moduli given by (37) can be called isotropic, since $A_{ijk\ell}$ does not depend on the choice of orthogonal coordinate system and is determined by one eigenvalue λ_1 and in this case the stress is $\sigma_{ij} = \lambda_1 \varepsilon_{ij}$. Materials that are traditionally referred to as isotropic form a special case of materials of the type (38).

Because the coordinate system is arbitrary, we will assume that the coordinates are along the principal axes of the tensors t_{ij11} and t_{ij12}. We denote the principal values of these tensors by α, β, γ; $\alpha/\sqrt{2}$, $\beta/\sqrt{2}$, $\gamma/\sqrt{2}$, respectively. Then condition (40) reduces to

$$\alpha^2 + \beta^2 + \gamma^2 = 1. \tag{41}$$

Now the elastic moduli (38) and (39) can be written as

$$A_{ijkl} = (\lambda_1 - \lambda_2)(\alpha \delta_{i1} \delta_{j1} + \beta \delta_{i2} \delta_{j2} + \gamma \delta_{i3} \delta_{j3}) \times \times (\alpha \delta_{k1} \delta_{l1} + \beta \delta_{k2} \delta_{l2} + \gamma \delta_{k3} \delta_{l3}) + \lambda_2 \delta_{ijkl}; \tag{42}$$

$$A_{ijkl} = \lambda_1 \delta_{ijkl} - (\lambda_1 - \lambda_6)(\alpha \delta_{i1} \delta_{j1} + \beta \delta_{i2} \delta_{j2} + + \gamma \delta_{i3} \delta_{j3})(\alpha \delta_{k1} \delta_{l1} + \beta \delta_{k2} \delta_{l2} + \gamma \delta_{k3} \delta_{l3}). \tag{43}$$

Obviously materials of the type (42) and (43) are characterized by four parameters: two eigenvalues and two of the parameters in (41). The difference between (43) and (42) is that in (43) there is a minus sign in front of $(\lambda_1 - \lambda_6)$. This material has a different internal structure.

If we assume the tensors t_{ij11} and t_{ij12} are spherical, i.e., we put $\alpha = \beta = \gamma = +1/\sqrt{3}$, then (42) and (43) take the form

$$A_{ijkl} = (1/3)(\lambda_1 - \lambda_2) \delta_{ij} \delta_{kl} + \lambda_2 \delta_{ijkl}; \tag{44}$$

$$A_{ijkl} = \lambda_1 \delta_{ijkl} - (1/3)(\lambda_1 - \lambda_6) \delta_{ij} \delta_{kl}. \tag{45}$$

The materials (44) and (45) are isotropic in the sense that $A_{ijk\ell}$ does not depend on the coordinate system, but is determined by two eigenvalues. If in (44) we let $(\lambda_1 - \lambda_2)/3 = \lambda$, $\lambda_2 = 2\mu$, then we have the traditional notation for the moduli of elasticity of an isotropic material.

The materials (44) and (45) are often taken as one; this is relevant to the question of the limits to the Poisson coefficient ν [1, p. 114; 4, p. 25; 12, p. 100; 13, p. 117; 14, p. 256]. But they are qualitatively different materials, belonging to classes with different structural symbols {1, 5} and {5, 1}. For (44) and (45) the Poisson coefficients are:

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}} = -\frac{a_{2211}}{a_{1111}} = -\frac{\frac{1}{3}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)}{\frac{1}{3}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) + \frac{1}{\lambda_2}} = \frac{\lambda_1 - \lambda_2}{2\lambda_1 + \lambda_2}; \tag{46}$$

$$\nu = -\frac{-\frac{1}{3}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_6}\right)}{\frac{1}{\lambda_1}-\frac{1}{3}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_6}\right)} = -\frac{\lambda_1-\lambda_6}{\lambda_1+2\lambda_6}. \qquad (47)$$

Because $\lambda_1 > \lambda_2 > 0$ and $\lambda_1 < \lambda_6 > 0$, it is evident from (46) and (47) that the Poisson coefficient lies between the following limits for (44) and (45), respectively:

$$0 < \nu < 1/2; \qquad (48)$$

$$-1 < \nu < 0. \qquad (49)$$

Hence the material (44) is a traditionally isotropic material whose Poisson coefficient satisfies (48). The material (45) is qualitatively different: upon extension of a rod of this material in an arbitrary direction, its transverse dimensions increase. For a material of this type, the Poisson coefficient satisfies (49).

For (37) $\nu = 0$. The class of materials (37) in a sense lie between the class of materials (44), which contract in the transverse dimensions upon a longitudinal extension of a rod, and the class of materials (45), which expand in the transverse dimensions under the same conditions.

In many texts on the theory of elasticity [1, p. 114; 4, p. 25; 12, p. 100; 14, p. 256] it is stated that materials with a negative Poisson coefficient are not observed experimentally. In [5] it is suggested that one look for materials with negative ν by doing experiments at very low temperatures, near absolute zero, and also a citation to an experiment is given in which $\nu = -0.102$.

The examples presented here demonstrate the usefulness of the classification of elastic materials proposed in the present paper. In the future it will be necessary to study all 32 classes of elastic materials in more detail. In [15-17] an analogous approach was given to the study of the structure of the generalized Hooke's law. In [16] the eigentensors t_{ijpq} were constructed in general form depending on 15 arbitrary parameters.

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